A Graphical Method for Transforming Impedances and Reflectances Through a Lossy Twoport*

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Summary—**In this paper a simple projective geometric construction is applied to the problem of finding the input impedance of a linear lossy twoport junction for a given output port termination. The method, an extension of Bracewell's transducer diagram, is presented in both the impedance and reflectance planes.**

It is well known that the relationship between terminating and input impedances or reflectances of a linear twoport network is of the form

$$Z' = \frac{aZ + b}{cZ + d}, \qquad ad - bc \neq 0. \tag{1}$$

In projective geometry it is shown that (1) establishes a projective correspondence between the points of two lines and, furthermore, if three points and their transforms are known, the correspondence is uniquely determined [1]. Hence, if it is known that a certain straight line in the Z plane is transformed into a certain straight line in the Z' plane, it is possible to set up a projectivity to find the input impedance or reflectance of a network for a given terminating impedance or reflectance.

Such a procedure has been given by Bracewell [2] for lossless networks in the impedance plane, which have the imaginary axis as an invariant line. In this communication Bracewell's diagram will be extended to lossy networks in the impedance and reflectance planes.

I. Impedance Plane

The effect of a lossy twoport, represented by (1), on a system of constant phase lines in the Z plane and constant magnitude circles in the Z plane is shown in Fig. 1. The systems L and C in the Z plane are transformed into L' and C' in the Z' plane.

Both L' and C' contain a circle which degenerates to a straight line, L_0' and C_0' in Fig. 1. L_0' corresponds to a certain constant phase line L_0, in the Z plane. If the lines L_0 and L_0' are known, Bracewell's method may be used to determine the input impedance for a given terminating impedance. The existence of an invariant line in the lossy case was acknowledged by Bracewell.

The measurement part of the procedure consists of terminating the network in a short circuit, an open circuit, and one other reactance and measuring the corresponding input impedances. The three measured impedances are plotted in a rectangular Z' plane and a circle is drawn through the points. This circle, shown as

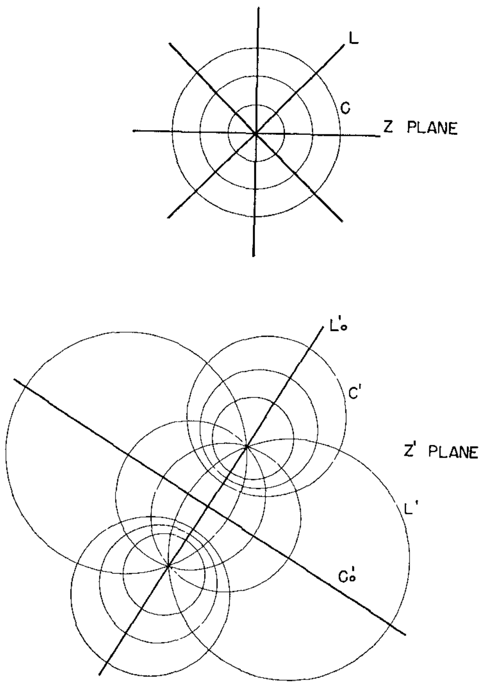

Fig. 1—Effect of a bilinear transformation on constant phase lines and constant magnitude circles.

L_j' in Fig. 2, is the transformed imaginary axis of the Z plane. The line through $Z'(0)$ and $Z'(\infty)$ is the Z'-plane image of the invariant constant phase line in the Z plane. The constant phase line which makes the same angle with the imaginary axis in the Z plane as L_0' makes with L_j' in the Z' plane is L_0.

The next step is to superimpose the Z and Z' planes such that L_0 and L_0' intersect in a point whose coordinates in the Z and Z' planes are paired by the transformation. The angle between L_0 and L_0' is arbitrary. In Fig. 2 the origin of the Z plane and its transform have been superimposed.

The third reactive termination is used to find the center of perspectivity as follows: A circle through this impedance, labelled Z_1 in Fig. 2, with center at the Z-plane origin cuts L_0 and the imaginary axis orthogonally. Referring to Fig. 2, this circle is labelled C_1 in the Z plane and its image in the Z' plane is labelled C_1'. The center of C_1' is located at the intersection of the tangent to L_j' at Z_1' and L_0'. The center of the perspectivity P, lies at the intersection of the lines A_1A_1' and A_2A_2'. It is essential that these lines be drawn through points which are paired by the transformation. These points are easily identified by the fact that the angle between the imaginary axis in the Z plane and L_0 (which

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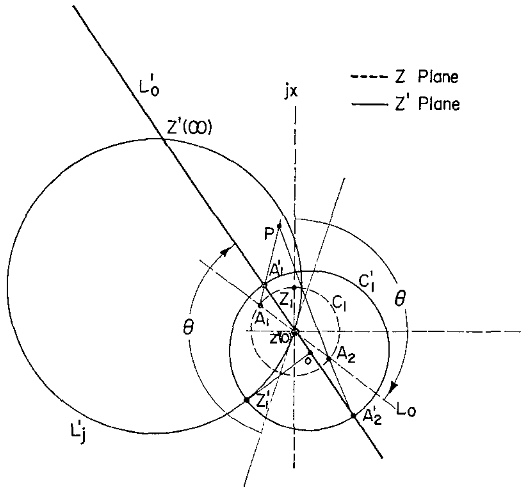

Fig. 2—Construction of center of perspectivity in the impedance plane.

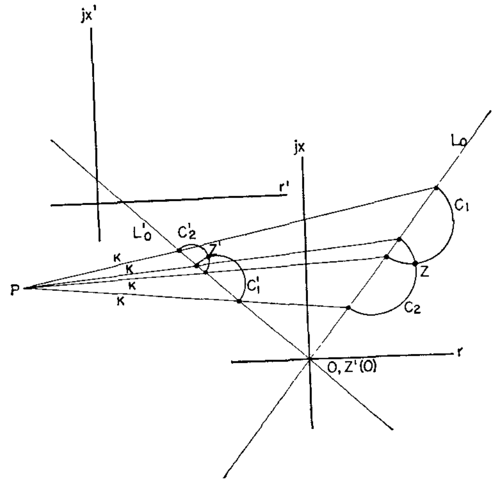

Fig. 3—Impedance transformation through a lossy twoport.

passes through A_2) must be equal to the angle between the transformed imaginary axis L_j', and L_0' (which passes through A_2') in the same sense. The remainder of the procedure is identical with Bracewell's method. It is included here for completeness.

The construction to determine the input impedance for an arbitrary termination is shown in Fig. 3. The preliminary constructions to find the center of perspectivity have been omitted for clarity. The two circles C_1 and C_2, with centers on L_0, are drawn to intersect in the point representing the termination of the network in the Z coordinate system. The lines of a perspectivity K, from the center of perspectivity P, to the intersections of C_1 and C_2 with L_0 produce a range of points on L_0'. The intersection of the circles C_1' and C_2', centered on L_0', on the same side relative to L_0' as is the termination to L_0, gives the input impedance to the network in the Z' coordinate system.

Another method for transforming impedances through lossy networks was given by Gemmel [4]. Gemmel's procedure uses three perspectivities and thereby eliminates the necessity for measuring the angle θ and causing L_0 and L_0' to intersect in corresponding points.

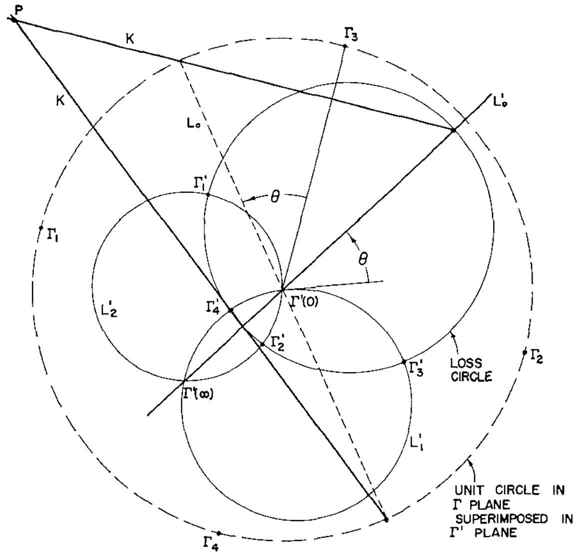

Fig. 4—Construction of center of perspectivity in the reflection coefficient plane.

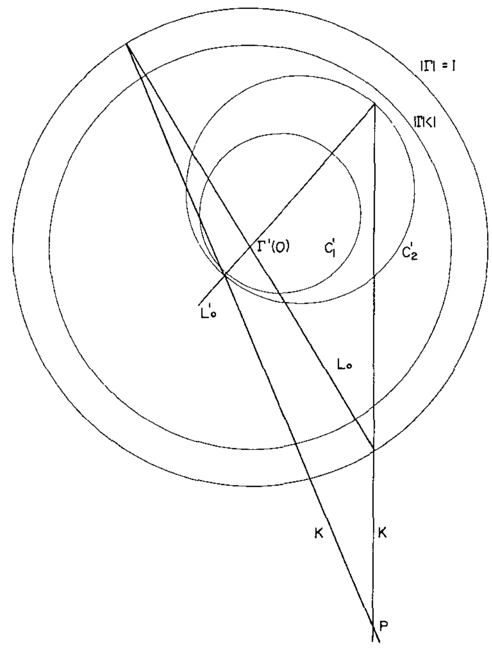

Fig. 5—Correcting a loss circle obtained with a lossy short.

II. Reflection Coefficient Plane

The two-port transformation in the reflection coefficient plane is also of the form of (1). Fig. 1 may be considered to be configurations in the Γ and Γ' planes. In the following development it will be assumed that the network is terminated in a sliding short.

The procedure for determining the invariant lines in the Γ and Γ' planes is very similar to that given above for the impedance plane. The essential difference is that the data points now lie on a circle of the system C' of Fig. 1.

The measurement part of the procedure consists of terminating the network in four reflectances which are separated by 90° on the unit circle in the Γ plane. Although three points are sufficient, four points simplify the constructions.

The four input reflectances are plotted in a Γ' plane and a circle is drawn through them (the loss circle). Two circles, L_1' and L_2', orthogonal to the loss circle, are drawn through the images of the diametrically paired points as shown in Fig. 4. The intersection of the orthogonal circles within the loss circle is $\Gamma'(0)$ and the external intersection is $\Gamma'(\infty)$. The line through these two points is L_0'. The angle between L_0' and any one of the orthogonal circles is equal to the angle between the corresponding constant phase line and the invariant line L_0, in the Γ plane (in the same sense).

As in the impedance plane, the Γ' plane is superimposed on the Γ plane such that L_0 and L_0' intersect in a pair of corresponding points. The center of perspectivity, P, is the intersection of the two lines, K, through the corresponding intersections of L_0 and the unit circle and L_0' and the loss circle. The procedure for finding the input reflectance for a given termination is identical with that given above for the impedance plane.

An interesting application of the method presented above is the graphical solution of the problem of correcting a loss circle obtained with a lossy sliding short whose locus is a circle concentric with the unit circle. This problem was considered analytically by Mathis [5].

The graphical solution is shown in Fig. 5. The preliminary constructions have been omitted for clarity. C_1' is the uncorrected loss circle. Since the unit circle is concentric with the locus of the sliding short, the corrected loss circle C_2' has its center on L_0' and passes through the intersections of the lines of the perspectivity K, with L_0'.

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